
On Guichard's nets and Cyclic systems

U. Hertrich-Jeromin*, E.-H. Tjaden and M.T. Zürcher

Dept Math & Stat, GANG, University of Massachusetts, Amherst, MA 01003

Dept Mathematics, SFB288, Technical University Berlin, D-10623 Berlin

Dept Mathematics, ETH-Zentrum, CH-8092 Zürich

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Summary. In the first part, we give a self contained introduction to the theory of cyclic systems in n -dimensional space which can be considered as immersions into certain Grassmannians. We show how the (metric) geometries on spaces of constant curvature arise as subgeometries of Möbius geometry which provides a slightly new viewpoint. In the second part we characterize Guichard nets which are given by cyclic systems as being Möbius equivalent to 1-parameter families of linear Weingarten surfaces. This provides a new method to study families of parallel Weingarten surfaces in space forms. In particular, analogs of Bonnet's theorem on parallel constant mean curvature surfaces can be easily obtained in this setting.

Introduction

Guichard nets were first mentioned by Guichard [3] who considered them as a 3-dimensional analog of isothermic nets [8]. Recently, the first author of the present paper discovered a close relation to the theory of conformally flat hypersurfaces in 4-space: any conformally flat hypersurface (in \mathbb{R}^4) carries curvature line coordinates which satisfy the Guichard condition [5]. In this sense, conformally flat hypersurfaces might be considered as “isothermic hypersurfaces¹⁾”. It seems remarkable that Guichard already introduces the “spectral parameter” which is used in the integrable system approach to conformally flat hypersurfaces [4]. In [5] it was also shown how Guichard nets in 3-space — now considered as special triply orthogonal systems of surfaces — do correspond to 3-dimensional

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¹⁾ It is still not clear whether the existence of Guichard curvature line coordinates *characterizes* conformally flat hypersurfaces.

conformally flat hypersurfaces: for certain, very special kinds of Guichard nets it was possible to characterize the corresponding conformally flat hypersurfaces geometrically. All of these “special” Guichard nets consist of a 1-parameter family of spheres and two 1-parameter families of channel surfaces, any two surfaces of different families intersecting orthogonally along a curvature line. On channel surfaces, one family of curvature lines consists of circles — thus, this type of Guichard nets can be considered as cyclic systems, i.e. as given by a 2-parameter family of circles which have a 1-parameter family of orthogonal surfaces. This was the observation initiating the present paper: we addressed the problem of classifying all Guichard nets which come from cyclic systems.

In the first part of this paper we will give a comprehensive introduction to the theory of cyclic systems. Even though cyclic systems and Möbius geometry are very well introduced in the classical literature (see [2] or [1]) it seemed worth to present this introduction here for two reasons: first, we are going to use Cartan’s method of moving frames which not only allows a modern formulation of the presented theory but also provides the tools to characterize the spaces of m -dimensional spheres in n -space as certain Grassmannians. Thus, the structure of the spaces of m -spheres becomes very lucid. And, second, we are going to present a different approach to Möbius geometry by introducing it as a supergeometry of the “metric” geometries of certain spaces of constant curvature. This new viewpoint in Möbius geometry allows to consider geometric problems in all spaces of constant curvature simultaneously — as we will learn in the second part of the present paper, by discussing Weingarten surfaces and generalizations of Bonnet’s theorem on parallel constant mean curvature surfaces in space forms. In the concluding section of the first part we will leave the n -dimensional setting and shortly discuss those basic facts in the theory of triply orthogonal systems in 3-space which we will need for our discussions on Guichard’s nets: systems of three 1-parameter families of surfaces such that any two surfaces from different families intersect orthogonally [8].

In the second part we will present a characterization for Guichard’s nets which come from cyclic systems — for “cyclic Guichard nets”. These turn out to be Möbius equivalent to 1-parameter families of parallel Weingarten surfaces in space forms, the Gauß and mean curvatures of all surfaces satisfying an affine relation. This is where the relation of Möbius geometry and its metric subgeometries becomes significant. We use this relation to present a way how various analogs of Bonnet’s theorem on parallel constant mean curvature surfaces in Euclidean space can be obtained (compare [7]). Considering a family of parallel linear Weingarten surfaces in a space form as a cyclic Guichard net the family is naturally parametrized by an elliptic function. We present relations between the properties of this elliptic function and the geometry of the family of parallel Weingarten surfaces — in particular, we relate the function’s branch points to surfaces of constant mean or constant Gauß curvature occurring in the family.

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Part I: Cyclic Systems

The main goal of this first part of the present paper is to provide a modern approach to the theory of cyclic systems — even though many of the results presented may be found in the classical literature (see for example Coolidge's excellent book [2]) it seemed worth not only to generalize the theory to n -dimensional space but also to point out the relations to symmetric spaces: circles and, more generally, m -spheres can be considered as elements in certain symmetric spaces. The theory of cyclic systems belongs into the context of Möbius geometry. Consequently, we give a short introduction to Möbius geometry — which is slightly different from the classical approach: instead of considering Euclidean space or the (metric or conformal) n -sphere as the underlying space we introduce Möbius geometry on spaces of constant curvature. This approach allows us to consider all metric geometries simultaneously as subgeometries of Möbius geometry — which will turn out useful in the second part of the paper. We conclude the first part with some considerations on triply orthogonal systems — again, slightly different from the classical approach (see for example [8]) since we will work in spaces of constant curvature instead of Euclidean space.

1. The metric geometries

Later, we are going to introduce Möbius geometry as a supergeometry of all the geometries given by the isometry groups on the spaces of constant curvature $k \in \mathbb{R}$. For this purpose we define quadric models for the spaces of constant curvature that will allow us to consider all constant curvature spaces simultaneously: let \mathbb{R}_1^{n+2} denote the $(n+2)$ -dimensional Minkowski space equipped with a Lorentz scalar product $\langle \cdot, \cdot \rangle$ of signature $(+, \dots, +, -)$ and

$$L^{n+1} := \{v \in \mathbb{R}_1^{n+2} \mid \langle v, v \rangle = 0\}$$

its light cone.

Lemma (spaces of constant curvature). *For any $n_k \in \mathbb{R}_1^{n+2} \setminus \{0\}$, the intersection*

$$Q_k^n := \{v \in L^{n+1} \mid \langle v, n_k \rangle = 1\}$$

of the light cone with the affine hyperplane $\langle v, n_k \rangle = 1$ is a Riemannian space of constant sectional curvature $k = -\langle n_k, n_k \rangle$.

In case $k \neq 0$ it is immediately clear that Q_k^n has constant sectional curvature: restricting our attention to the hyperplane $\langle v, n_k \rangle = 1$ we see that Q_k^n is a round sphere of radius $\frac{1}{\sqrt{k}}$ or a two-sheeted hyperboloid of “radius” $\frac{1}{\sqrt{-k}}$ in $(n+1)$ -dimensional Minkowski space. In case $k = 0$ we choose a point $p_0 \in Q_0^n$ and orthogonally decompose the Minkowski space $\mathbb{R}_1^{n+2} = \mathbb{R}^n \oplus \text{span}\{p_0, n_0\}$ — note that the plane $\text{span}\{p_0, n_0\}$ has signature $(+, -)$. Then, the map $x \in \mathbb{R}^n \mapsto x + p_0 - \frac{1}{2}|x|^2 n_0 \in Q_0^n$ is an isometry²⁾. Thus, for $k \geq 0$ we

²⁾ In fact, this isometry is the parametrization often used in the classical introductions to Möbius geometry (cf. [1]).

got models for the space forms — complete and simply connected — of curvature k whilst in case $k < 0$ we got spaces which are composed of two copies of a space form of sectional curvature k .

Before discussing the isometry groups of our spaces of constant curvature we have to understand the relations between these spaces: consider two quadrics Q_k^n and $Q_{\tilde{k}}^n$ of constant curvatures k and \tilde{k} . Then, we can map one onto the other by means of a rescaling $p \in Q_k^n \mapsto \tilde{p} = \pm e^u p \in Q_{\tilde{k}}^n$ with a suitable function u — note that this map lacks to be defined for the asymptotic directions $p \perp n_{\tilde{k}}$, the “infinity boundary”, of $Q_{\tilde{k}}^n$. Since $\langle p, p \rangle \equiv 0$ we have

$$\langle d(e^u p), d(e^u p) \rangle = e^{2u} \langle dp, dp \rangle,$$

i.e. these maps between our spaces of constant curvature are conformal. Moreover, in case of the map $Q_1 \rightarrow Q_0$ we obtain the classical stereographic projection after “correctly” identifying $Q_1^n \cong S^n$ and $Q_0^n \cong \mathbb{R}^n$ which convinces us of the following

Definition (generalized stereographic projections). *The central projection from one quadric Q_k^n onto another $Q_{\tilde{k}}^n$ is called a generalized stereographic projection.*

Clearly, the Lorentz transformations $F \in O_1(n+2)$ which fix n_k act as isometries on Q_k^n . Let us try to understand the converse: given an isometry $f : Q_k^n \rightarrow Q_{\tilde{k}}^n$ we may define a Lorentz transformation $F : \mathbb{R}_1^{n+2} \rightarrow \mathbb{R}_1^{n+2}$ to linearly approximate the isometry around one point $p \in Q_k^n$:

$$Fn_k = n_k, \quad Fp = f(p), \quad F|_{T_p Q_k^n} = d_p f$$

— note, that $T_p Q_k^n = \{p, n_k\}^\perp$. On space forms, isometries are uniquely determined by their behaviour at one point. Consequently, for $k \geq 0$, the group of isometries is identical with the group of those Lorentz transformations that fix n_k . For $k < 0$, the situation is a bit more complicated — in this case the quadrics Q_k^n are not connected. To learn the specialty of those transformations coming from Lorentz transformations of the ambient Minkowski space we project such a hyperbolic quadric Q_k^n , $k < 0$, stereographically into a sphere Q_1 — this way we make the infinity boundary between the two connected components of Q_k^n “visible”. An isometry which comes from a Lorentz transformation extends smoothly through the infinity boundary³⁾ $\langle p, n_k \rangle = 0$:

Definition (proper isometries). *The group of proper isometries of a quadric Q_k^n of constant curvature k is identical with the group of Lorentz transformations that fix the global normal vector n_k of Q_k^n :*

$$Isom(Q_k^n) = \{F \in O_1(n+2) \mid Fn_k = n_k\}.$$

³⁾ On Q_1 , an isometry of Q_k^n appears as a conformal transformation — composed of an isometry and two stereographic projections.

For the rest of this paper we will refer to “proper isometries” simply as “isometries”. Note, that in case $k < 0$ the space of proper isometries is still not the space of hyperbolic motions but a twofold covering of it: there are proper isometries which interchange the two connected components of a hyperbolic quadric Q_k^n .

2. Spheres and Circles

In order to introduce Möbius geometry as a supergeometry of all the metric geometries just discussed we also have to understand the space of spheres in those spaces Q_k^n . To start with, let us consider hyperspheres as totally umbilic, “conformally connected⁴⁾” hypersurfaces in Q_k^n . Given a hypersphere S we may write the umbilicity condition $dn + hdp = 0$ where p varies on S , n is a unit normal field to $S \subset Q_k^n$ and h denotes the (constant) mean curvature of S with respect to n . This equation integrates to $n + hp = s$ where $s \in \mathbb{R}_1^{n+2}$ is a constant vector. Since $n|_p \in T_p Q_k^n = \{p, n_k\}^\perp$ we find that $s \perp p$ is a unit vector and, moreover, we obtain the sphere's mean curvature as

$$h = \langle s, n_k \rangle.$$

At this point, it becomes clear that the vector s in fact *characterizes* the (oriented) sphere S : if $p \perp s$ then $s - hp \in T_p Q_k^n$ and consequently, p is a point on a totally umbilic hypersurface with (constant) mean curvature $h = \langle s, n_k \rangle$ and unit normal vector $n|_p = s - hp$ in p . Totally umbilic hypersurfaces with vanishing mean curvature are usually called hyperplanes — we obtain the hyperplanes in Q_k^n as special hyperspheres: as long as we are interested in Möbius geometry rather than in any of its metric subgeometries it is convenient not to distinguish them.

Since the equation $\langle p, s \rangle = 0$ encoding incidence of a point $p \in Q_k^n$ and a sphere $s \in S_1^{n+1}$ is obviously independent of the scaling of p , spheres are mapped to spheres by the generalized stereographic projections — actually, the notion of a “sphere” does not depend on the metric of a space but on its conformal class only. As a fundamental invariant in Möbius geometry also the intersection angle of two (oriented) spheres can be nicely described in this model: given two spheres intersecting in a point p , $s = n + hp$, $\tilde{s} = \tilde{n} + \tilde{h}p \in S_1^{n+1}$, their intersection angle is given by

$$\langle n, \tilde{n} \rangle = \langle s, \tilde{s} \rangle.$$

Lemma (oriented spheres). *The space of oriented hyperspheres in any of the spaces Q_k^n of constant curvature can be canonically identified with the Lorentz sphere $S_1^{n+1} \subset \mathbb{R}_1^{n+2}$; hyperplanes in a Q_k^n may be distinguished by the vanishing of the mean curvature:*

$$\langle n_k, s \rangle = 0.$$

⁴⁾ Again, there arise problems with the hyperbolic spaces: here, spheres can be “connected through the infinity boundary”, i.e. a sphere may consist of two pieces which can be smoothly glued together by adding a submanifold of the infinity boundary.

A point $p \in Q_k^n$ lies on a sphere $s \in S_1^{n+1}$ if and only if

$$\langle p, s \rangle = 0.$$

Two spheres $s, \tilde{s} \in S_1^{n+1}$ intersect orthogonally if and only if

$$\langle s, \tilde{s} \rangle = 0.$$

From the previous discussions it also becomes clear that two hyperspheres $s = n|_p + hp$ and $\tilde{s} = n|_p + \tilde{h}p = s + (\tilde{h} - h)p$ have the same tangent planes in their common point p :

Definition (parabolic sphere pencil). *The 1-parameter family $s + \mathbb{R}p$ of all oriented hyperspheres sharing tangent planes (and orientation) in one point p is called a parabolic sphere pencil — or an (oriented) hypersurface element.*

Later on, we will not only be interested in hyperspheres but in lower dimensional spheres, too. Especially, since we are going to discuss cyclic systems, we will be interested in circles: 1-dimensional spheres. Similar to the hypersphere case, we may consider m -dimensional spheres in any of the spaces Q_k^n ($m < n$) as totally umbilic submanifolds of dimension m — which can be obtained intersecting $n - m$ totally umbilic (orthogonal) hypersurfaces: hyperspheres. Thus, the points p of an m -sphere are given by

$$p \in L^{n+1} \cap \{s_1, \dots, s_{n-m}\}^\perp$$

where $s_1, \dots, s_{n-m} \in S_1^{n+1}$. Obviously, this characterization does not depend on the choice of hyperspheres but only on the $(n - m)$ -plane spanned by the s_i :

Theorem (m -spheres). *The space of (non oriented) m -spheres in any of the spaces Q_k^n of constant curvature can be canonically identified with the Grassmannian*

$$G_+(n - m, m + 2) = \frac{O_1(n + 2)}{O(n - m) \times O_1(m + 2)}$$

of spacelike $(n - m)$ -planes of the Minkowski space \mathbb{R}_1^{n+2} ; m -planes in a Q_k^n are given by those spacelike $(n - m)$ -planes in \mathbb{R}_1^{n+2} which are perpendicular to n_k .

Note that this theorem complements very well our previous statement on oriented $(n - 1)$ -spheres: the Lorentz sphere S_1^{n+1} is a double cover of the Grassmannian $G_+(1, n + 1)$ of spacelike lines in \mathbb{R}_1^{n+2} — the two possible orientations of a spacelike line in \mathbb{R}_1^{n+2} can be interpreted as the orientations of the corresponding hypersphere. Similarly, m -spheres could be oriented by considering the two possible orientations on spacelike $(n - m)$ -planes.

To conclude this section we will discuss 1-spheres — commonly called “circles” — a bit more comprehensively: let $c = \text{span}\{s_1, \dots, s_{n-1}\} \in G_+(n - 1, 3)$ denote a circle. The points on c are given by lightlike lines⁵⁾ in c^\perp . So, we may

⁵⁾ Note, that for the moment we want our arguments to be independent of the ambient constant curvature spaces Q_k^n — we want to obtain *Möbius geometric* notions.

obtain a parametrization of c by choosing a basis for c^\perp : to fit the geometric situation best we choose a *pseudo orthonormal* basis (s, p, \hat{p}) , i.e. $s \in S_1^{n+1}$ is an (oriented) sphere and $p, \hat{p} \in L^{n+1}$ are two points on s ($\langle s, p \rangle = \langle s, \hat{p} \rangle = 0$) with $\langle p, \hat{p} \rangle = 1$. Since all the spheres s_i intersect s orthogonally and contain p as well as \hat{p} ($s_i \perp s, p, \hat{p}$) our circle c is exactly that circle intersecting the sphere s orthogonally in the two points p and \hat{p} : this yields a description for circles which complements our first description in some sense.

Given a circle c in this complementary description — by two points p, \hat{p} on a sphere s — its most general “arc length” parametrization into the light cone⁶⁾ is given by

$$t \mapsto p_t = \frac{1}{g'}(g s + p - \frac{1}{2}g^2 \hat{p}) \in L^{n+1} \cap c^\perp \quad (1)$$

where $g = g(t)$ denotes *any* function of t . To understand the geometry of this parametrization we fix a projective scale — three points — on the circle: in our setting a somehow canonical choice would be $p = p_t|_{g=0}$, $\hat{p} = p_t|_{g=\infty}$ and $p_t|_{g=1} = s + p - \frac{1}{2}\hat{p}$. With this choice, $g(t)$ is the cross ratio⁷⁾ of the three scale points and p_t :

$$g(t) = R(p, p_t, \hat{p}, p_t|_{g=1}) = \sqrt{\frac{\langle p, g s + p - \frac{1}{2}g^2 \hat{p} \rangle \langle \hat{p}, s + p - \frac{1}{2}\hat{p} \rangle}{\langle g s + p - \frac{1}{2}g^2 \hat{p}, \hat{p} \rangle \langle s + p - \frac{1}{2}\hat{p}, p \rangle}}. \quad (2)$$

Note, that this cross ratio does not depend on the scaling of the points in the light cone: it is a conformal invariant, just like the intersection angle of two hyperspheres.

An example of a circle which will become important in the second part is a straight line in any of the constant curvature spaces Q_k^n . Here, we may choose the sphere s in our parametrization as a plane, i.e. $\langle s, n_k \rangle = 0$. Since we are considering a straight line we also have $\langle s_i, n_k \rangle = 0$ and, consequently, $n_k \in \text{span}\{p, \hat{p}\}$. Now, assuming p to be a point in Q_k^n , we find $n_k = -\frac{k}{2}p + \hat{p}$. For the function g , this yields the differential equation $1 = \langle p_t, n_k \rangle = \frac{1}{g'}(1 + \frac{k}{4}g^2)$; its solutions — we fix the initial value $g(0) = 0$, i.e. $p_0 = p$ —

$$g(t) = \begin{cases} \frac{2 \sinh(\sqrt{-k}t)}{\sqrt{-k}(1 + \cosh(\sqrt{-k}t))} & \text{for } k < 0 \\ t & \text{for } k = 0 \\ \frac{2 \sin(\sqrt{k}t)}{\sqrt{k}(1 + \cos(\sqrt{k}t))} & \text{for } k > 0 \end{cases}$$

provide the arc length parametrizations for straight lines in Q_k^n :

$$p_t = \begin{cases} -\frac{1}{k} n_k + \frac{\sinh(\sqrt{-k}t) s + \cosh(\sqrt{-k}t) (\frac{\sqrt{-k}}{2}p - \frac{1}{\sqrt{-k}}\hat{p})}{\sqrt{-k}} & \text{for } k < 0 \\ t s + p - \frac{1}{2}t^2 \hat{p} & \text{for } k = 0 \\ -\frac{1}{k} n_k + \frac{\sin(\sqrt{k}t) s + \cos(\sqrt{k}t) (\frac{\sqrt{k}}{2}p + \frac{1}{\sqrt{k}}\hat{p})}{\sqrt{k}} & \text{for } k > 0 \end{cases} \quad (3)$$

⁶⁾ Again, note that we do *not* require p_t to take values in any of our constant curvature spaces — this will turn out convenient in the second part since we are going to study special coordinate systems there. To make the parametrization take values in a space of constant curvature p_t would have to be suitably rescaled.

⁷⁾ In this paper we consider the cross ratio an invariant of a quadrilateral rather than of a point pair: our cross ratio differs from the classical one by a transposition of the points.

Note, that p_t never reaches the infinity boundary of Q_k^n since $\langle p_t, n_k \rangle \equiv 1 \neq 0$. In case $k < 0$, this means that p_t only parametrizes half of a straight line, that component which belongs to $p = p_0$.

At this point we are prepared to introduce

3. Möbius geometry as a supergeometry

of the metric geometries on the spaces Q_k^n of constant curvature. To this extend we have to define the group of Möbius transformations on Q_k^n :

Definition (Möbius group). *A transformation⁸⁾ of Q_k^n which maps hyperspheres to hyperspheres is called a Möbius transformation of Q_k^n . The group formed by all Möbius transformations is called the Möbius group.*

As we noticed earlier, the stereographic projections map spheres in any Q_k^n to spheres in any Q_k^n . Since a Lorentz transformation $F \in O_1(n+2)$ induces an isometry $F : Q_k^n \rightarrow F(Q_k^n)$ it clearly maps spheres in Q_k^n to spheres in $F(Q_k^n)$, a quadric of the same constant curvature k but, generally, different from Q_k^n . Thus, by composing F with a suitable (unique) stereographic projection we obtain a Möbius transformation of Q_k^n . In fact, *all* Möbius transformations of a quadric Q_k^n of constant curvature can be obtained this way:

Lemma (Möbius transformations). *Any Möbius transformation μ of Q_k^n is the composition $\mu = \sigma_F \circ F$ of a Lorentz transformation $F \in O_1(n+2)$ with the (unique) stereographic projection $\sigma_F : F(Q_k^n) \rightarrow Q_k^n$.*

Before attacking the proof of this lemma let us state some facts: first, it becomes clear that Möbius transformations are conformal, i.e. they preserve intersection angles between spheres. And second, the interplay of a Möbius transformation with the infinity boundary of Q_k^n in case $k \leq 0$ becomes clear: the stereographic projection σ_F lacks to be defined for the points $p \in Q_k^n$ mapped to the infinity boundary n_k^\perp of Q_k^n . Similarly, the infinity boundary of Q_k^n will generally be mapped to a finite region — to a point for $k = 0$ and to a hypersphere in case $k < 0$.

In the above construction of a Möbius transformation, by composing a Lorentz transformation with a stereographic projection, the (uniquely determined) stereographic projection was only needed to adjust the scaling of points correctly. If, for a moment, we identify all the quadrics of constant curvature by identifying points with lightlike lines⁹⁾, $p \leftrightarrow \mathbb{R} \cdot v$ where $v \in L^{n+2}$, we can

⁸⁾ Here, we use the term “transformation of Q_k^n ” in a slightly generalized sense: we allow a Möbius transformation to miss the preimage and the image of the infinity boundary of Q_k^n — two points for $k = 0$ and two spheres for $k < 0$.

⁹⁾ This is the only point where we really adapt the classical viewpoint: classically, an (absolute) quadric in projective $(n+1)$ -space $\mathbb{R}P^{n+1}$, the “conformal n -sphere”, is considered as the underlying space for Möbius geometry — the Minkowski space \mathbb{R}_1^{n+2} becomes the space of homogeneous coordinates of $\mathbb{R}P^{n+1}$ the Lorentz product being fixed (up to scaling) by the conformal n -sphere as absolute quadric. In this model, m -spheres are identified with $(n-m-1)$ -planes by polarity.

identify a Möbius transformation $\mu = \sigma_F \circ F$ with the corresponding Lorentz transformation $F \in O_1(n+2)$.

Now, that we do not have to care about the “proper” scaling of points any more the proof of our lemma becomes easy: preserving hyperspheres a Möbius transformation will also preserve m -spheres since those can be obtained as intersections of hyperspheres. Consequently, a Möbius transformation naturally extends to a linear transformation of the Minkowski space \mathbb{R}_1^{n+2} : it maps the spaces $G_+(n-m, m+2)$ of m -spheres onto themselves. But, it is well known that linear transformations of \mathbb{R}_1^{n+2} which preserve the light cone are real multiples of Lorentz transformations. Thus, given a Möbius transformation $\mu : Q_k^n \rightarrow Q_k^n$, there is a Lorentz transformation (unique up to sign) $F \in O_1(n+2)$ such that $\mu = \sigma_F \circ F$ — this proves the lemma. Moreover,

Lemma (Möbius group). *The group of Lorentz transformations is a (trivial) double covering of the Möbius group:*

$$\text{Möb}(Q_k^n) \times \{\pm 1\} \cong O_1(n+2).$$

From the way how we introduced the metric geometries it now is immediately clear that Möbius geometry is a supergeometry of the metric geometries¹⁰:

Theorem (metric subgeometries). *The geometries of the groups of motions on the quadrics Q_k^n are subgeometries of the Möbius geometry on Q_k^n .*

4. Envelopes

Given an immersion $f : M^{n-1} \rightarrow Q_k^n$ with unit normal field $n : M^{n-1} \rightarrow S_1^{n+1}$ the normal field may be reinterpreted as a sphere congruence — a 2-parameter family of spheres — according to our identification of spheres in Q_k^n with unit vectors in \mathbb{R}_1^{n+2} . Since $\langle f, n \rangle = 0$ any point $f(p)$ lies on the corresponding sphere $n(p)$. Moreover, f and $n(p)$ have first order contact — f “touches” $n(p)$ — in $f(p)$ since $\langle d_p f, n(p) \rangle = 0$: $n(p)$ can be considered a common normal of $f(M)$ and $n(p)$ in $f(p)$. Note the different interpretations for $n(p)$ — we will use this ambiguity of the geometric meaning of $n(p)$ repeatedly. Any other sphere $s \in S_1^{n+1}$ touching $f(M)$ in $f(p)$ lies in one of the parabolic sphere pencils $\pm(n(p) + \mathbb{R}f(p))$, these two sphere pencils are *characterized* by the equations $\langle s, f(p) \rangle = 0$ and $\langle s, d_p f \rangle = 0$:

Definition (envelope). *An immersion $f : M^{n-1} \rightarrow Q_k^n$ is said to envelope a sphere congruence $s : M^{n-1} \rightarrow S_1^{n+1}$ if each sphere $s(p)$ touches $f(M)$ in $f(p)$:*

$$\langle s(p), f(p) \rangle = 0 \quad \text{and} \quad \langle s(p), d_p f \rangle = 0. \quad (4)$$

If we consider the immersion f enveloping a sphere congruence s to take values in the light cone, $f : M \rightarrow L^{n+1}$, rather than in one of the quadrics

¹⁰ In a similar way as Möbius geometry is a subgeometry of projective geometry (cf. Klein's Erlanger program [6]).

$Q_k^n \subset L^{n+1}$ then, the sphere congruence s can still be interpreted as a (unit) normal field of f according to (4): $s(p) \in T_{f(p)}L^{n+1} = f(p)^\perp$. On the other hand, since $\langle f, s \rangle \equiv 0$ and consequently $\langle f(p), d_p s \rangle = -\langle s(p), d_p f \rangle$, the immersion f can as well be interpreted as an (isotropic) normal field of the sphere congruence $s : M \rightarrow S_1^{n+1}$: starting from a hypersphere congruence s which induces a positive definite metric (i.e. $d_p s(T_p M) \in G_+(n-1, 3)$) we will be able to find *two* envelopes f and \hat{f} since the normal bundle of s has signature $(+, -)$. Geometrically, the points $f(p)$ and $\hat{f}(p)$ of the two envelopes are given as the intersection points of $s(p)$ and the circles $d_p s(T_p M)$. — Focussing on the geometry of a sphere congruence rather than on the geometry of one of its envelopes it will often be more useful not to scale the envelopes to lie in a quadric Q_k^n of constant curvature — as it might be as well if we are interested in conformal aspects of the hypersurface's geometry.

To reflect the multiple aspects of a sphere congruence and one of its envelopes by a more neutral term¹¹⁾ we give the following

Definition (strip). *A pair $(s, f) : M^{n-1} \rightarrow S_1^{n+1} \times L^{n+1}$ of smooth maps is called a strip if at least one, s or f , is an immersion with spacelike tangent planes and if*

$$\langle s, f \rangle = 0 \quad \text{and} \quad \langle s, df \rangle = 0.$$

To discuss the geometry of strips we will use Cartan's method of moving

5. Frames

Given a strip $(s, f) : M \rightarrow S_1^{n+1} \times L^{n+1}$ we extend it to a pseudo orthonormal frame $F := (s_1, \dots, s_{n-1}, s, f, \hat{f}) : M \rightarrow SO_1(n+2)$ into the Lorentz group. The corresponding connection form $\Phi = F^{-1}dF : TM \rightarrow \mathfrak{o}_1(n+2)$ will be of the form

$$\Phi = \begin{pmatrix} \omega & \eta \\ -\eta^* & \nu \end{pmatrix} \quad (5)$$

where $\eta : TM \rightarrow M(3 \times (n-1))$, $\omega : TM \rightarrow \mathfrak{o}(n-1)$ and $\nu : TM \rightarrow \mathfrak{o}_1(3)$ describe the derivative of the circle congruence $c := \text{span}\{s_1, \dots, s_{n-1}\}$ and the covariant derivatives on the vector bundles c and c^\perp , respectively. This splitting of the connection form Φ corresponds to the Cartan decomposition¹²⁾

$$\mathfrak{o}_1(n+2) = (\mathfrak{o}(n-1) \oplus \mathfrak{o}_1(3)) \oplus M(3 \times (n-1)) =: \mathfrak{k} \oplus \mathfrak{p} \quad (6)$$

of the Lie algebra $\mathfrak{o}_1(n+2)$ associated with the symmetric space $G_+(n-1, 3)$ of circles in Q_k^n .

¹¹⁾ Note, that in our definition of a strip we also get rid of any scaling requirements for the point map f — as it seems useful when interpreting it as a normal field of the sphere congruence.

¹²⁾ As we will see, circle congruences (and more generally, congruences of m -spheres) can be considered as immersions into the Grassmannian $G_+(n-1, 3)$ ($G_+(n-m, m+2)$). For that reason, we refer to the Cartan decomposition at this point.

First, let us have a closer look at

$$\nu = \begin{pmatrix} 0 & 0 & -\nu_s \\ \nu_s & \nu_f & 0 \\ 0 & 0 & -\nu_f \end{pmatrix} \quad (7)$$

where $\nu_s = \langle ds, \hat{f} \rangle$ and $\nu_f = \langle df, \hat{f} \rangle$. Since at least one, s or f , is assumed to be an immersion we may choose s_1, \dots, s_{n-1} to span the tangent space of either s or f — depending on which one is an immersion and on whether we are interested in the geometry of s or f :

Definition (adapted frames). A frame $(s_1, \dots, s_{n-1}, s, f, \hat{f})$ of a strip (s, f) is called *s-adapted* (or *f-adapted*) if s_1, \dots, s_{n-1} span the tangent planes of s (or f), i.e. if, in (7), $\nu_s = 0$ (or, $\nu_f = 0$).

In case of an s -adapted frame, \hat{f} describes the second envelope of the sphere congruence s and the tangent planes $d_p s(T_p M)$ of s define the congruence of circles orthogonal to s in its two envelopes f and \hat{f} . A special case occurs when s is a hyperplane congruence in Q_k^n : then, \hat{f} is the antipode hypersurface of f ($k > 0$), is the point at infinity ($k = 0$) or it is the reflection of f at the infinity hypersphere n_k ($k < 0$). In case of an f -adapted frame, \hat{f} will usually *not* be the second envelope of s — however, the circles $d_p f(T_p M)$ will still intersect the spheres $s(p)$ orthogonally in $f(p)$ and $\hat{f}(p)$. If, for example, $f : M \rightarrow Q_k^n$ then the circles $d_p f(T_p M)$ are straight lines in Q_k^n since $\langle df, n_k \rangle = 0$ and \hat{f} will be an envelope of s only if \hat{f} is a “parallel surface” of f , i.e. if $\langle s, n_k \rangle = \text{const.}$

According to the Cartan decomposition (6) of $\mathfrak{o}_1(n+2)$ the Maurer-Cartan equation¹³⁾ $d\Phi + \frac{1}{2}[\Phi \wedge \Phi] = 0$ splits into the Gauß-Ricci equations

$$d\omega + \omega \wedge \omega = \eta \wedge \eta^* \quad (8)$$

and

$$d\nu + \nu \wedge \nu = \eta^* \wedge \eta \quad (9)$$

and the Codazzi equation

$$d\eta + \omega \wedge \eta + \eta \wedge \nu = 0. \quad (10)$$

Since we will always work with adapted frames where ω describes the covariant derivative of $s(M)$ (resp $f(M)$) we will refer to (8) as the Gauß equation and

¹³⁾ Here, $[\Phi \wedge \Psi](x, y) := [\Phi(x), \Psi(y)] - [\Phi(y), \Psi(x)]$ where Φ and Ψ are Lie algebra valued 1-forms. In case of a matrix Lie algebra, $\Phi = (\varphi_{ij})$, (as in our case), where the Lie bracket becomes the commutator, we may write

$$[\Phi \wedge \Phi] = 2\Phi \wedge \Phi := 2 \sum_j \varphi_{ij} \wedge \varphi_{jk}.$$

to (9) as the Ricci equation. — From now on, let us assume the frame F to be either s -adapted or f -adapted. With the ansatz

$$\eta = \begin{pmatrix} -\eta_1 & \varphi_1 & \hat{\varphi}_1 \\ \vdots & \vdots & \vdots \\ -\eta_{n-1} & \varphi_{n-1} & \hat{\varphi}_{n-1} \end{pmatrix} \quad (11)$$

one of the scalar Ricci equations reads $\sum_j \eta_j \wedge \varphi_j = 0$ showing that the second fundamental form $\sum_j \eta_j \varphi_j$ of $s(M)$ with respect to the isotropic normal field f or of $f(M)$ with respect to the unit normal field s , respectively, is symmetric. Consequently, the tangential framing (s_1, \dots, s_{n-1}) can be chosen to simultaneously diagonalize the first and second fundamental forms.

Definition (curvature framing). *An (s - or f -) adapted framing is called a principal curvature framing of the strip (s, f) if its tangential part (s_1, \dots, s_{n-1}) diagonalizes the second fundamental form with respect to the induced metric:*

$$\eta_i \wedge \varphi_i = 0.$$

The directions orthogonal to the planes $\eta_i = 0$ (or $\varphi_i = 0$, respectively) are then called principal curvature directions of the strip.

Passing from an s -adapted to the “nearest” f -adapted principal curvature framing of the strip (s, f) — provided both, s as well as f , are immersions — via $s_i \mapsto s_i + u_i f$ with suitable functions u_i , the forms η_i and φ_i do not change. Consequently, the principal curvature directions do not change either.

Moreover, a “conformal deformation” $(s, f) \mapsto (s + hf, e^u f)$ of a strip (s, f) yields $\eta_i \mapsto (\eta_i - h\varphi_i)$ and $\varphi_i \mapsto e^u \varphi_i$ for any corresponding principal curvature framing. Hence, the principal curvature directions are also not effected by such a conformal deformation — passing from a strip (s, f) to the corresponding immersion $f : M \rightarrow Q_k^n$ into a quadric of constant curvature k with its unit normal field $n : M \rightarrow TQ_k^n$ we see that the principal curvature directions defined above are indeed the principal curvature directions in the classical sense. We summarize these results in a

Lemma. *The principal curvature directions of an immersion $f : M \rightarrow Q_k^n$ are conformally invariant, i.e. they coincide with the principal curvature directions of any strip $(s, e^u f) : M \rightarrow S_1^{n+1} \times L^{n+1}$. In particular, the principal curvature directions are invariant under the stereographic projections $Q_k^n \rightarrow Q_{\tilde{k}}^n$.*

Given an immersed sphere congruence $s : M \rightarrow S_1^{n+1}$ we have seen that the two isotropic normal fields in an s -adapted framing $F : M \rightarrow O_1(n+2)$ can be interpreted as the two envelopes of s . Assuming both, f as well as \hat{f} , to be immersed the principal curvature directions of (s, f) and (s, \hat{f}) will generally *not* coincide. If, however, they do we have $\eta_i \wedge \varphi_i = 0$ and $\eta_i \wedge \hat{\varphi}_i$ at the same time for any s -adapted principal curvature framing F . In this case we can arrange to have $\nu = 0$ by possibly rescaling $(f, \hat{f}) \mapsto (e^u f, e^{-u} \hat{f})$ the two isotropic normal fields: the vector bundle $c^\perp = \text{span}\{s, f, \hat{f}\}$ is flat.

Definition (Ribaucour sphere congruence). *An (immersed) sphere congruence $s : M \rightarrow S_1^{n+1}$ is called a Ribaucour sphere congruence if its normal bundle is flat: $\eta^* \wedge \eta = 0$. If its two envelopes f and \hat{f} are immersed, this means that their curvature directions do correspond.*

6. Cyclic systems

Let $F = (s_1, \dots, s_{n-1}, s, f, \hat{f}) : M^{n-1} \rightarrow O_1(n+2)$ denote an s -adapted framing of a strip $(s, f) : M^{n-1} \rightarrow S_1^{n+1} \times L^{n+1}$ where s is a Ribaucour sphere congruence and $f, \hat{f} : M \rightarrow L^{n+1}$ are *parallel* isotropic normal fields of s , i.e. $\nu = 0$. Then,

$$f_t := \frac{1}{g'(t)} \left(g(t) \cdot s + f - \frac{1}{2} g^2(t) \cdot \hat{f} \right) \quad (12)$$

will provide simultaneous arc length parametrizations (1) for all circles of the congruence $c = \text{span}\{s_1, \dots, s_{n-1}\} : M \rightarrow G_+(n-1, 3)$. Moreover, all circles intersect each hypersurface of the 1-parameter family $(f_t)_t$ orthogonally since

$$\left\langle \frac{\partial}{\partial t} f_t, df_t \right\rangle = 0 \quad (13)$$

— the circle congruence c is what is called a “cyclic system”:

Definition (cyclic system). *A circle congruence $c : M^{n-1} \rightarrow G_+(n-1, 3)$ is called a normal congruence of circles, or a cyclic system, if there is a 1-parameter family of hypersurfaces (in Q_k^n) orthogonal to all circles.*

Clearly, any immersed sphere congruence defines a circle congruence which has two orthogonal hypersurfaces — the two envelopes of the sphere congruence. Generally, these are the only hypersurfaces which are orthogonal to all circles of the congruence — as we will see: let $c : M^{n-1} \rightarrow G_+(n-1, 3)$ denote a circle congruence and let $F = (s_1, \dots, s_{n-1}, s, f, \hat{f}) : M \rightarrow O_1(n+2)$ be a pseudo orthonormal framing of c . Note, that F is not necessarily a framing of a strip, i.e. f might *not* describe an envelope of the sphere congruence s : this fact is responsible for the need to consider a more general form of the connection form (5) of F :

$$\nu = \begin{pmatrix} 0 & -\hat{\nu}_s & -\nu_s \\ \nu_s & \nu_f & 0 \\ \hat{\nu}_s & 0 & -\nu_f \end{pmatrix}.$$

With the ansatz (12), $g \equiv 1$ — but t now denoting a function on M^{n-1} , for the orthogonal hypersurfaces of the circle congruence the orthogonality condition (13) yields the following differential equation for $t : M^{n-1} \rightarrow \mathbb{R}$:

$$dt = \frac{1}{2} t^2 \nu_s + t \nu_f + \hat{\nu}_s. \quad (14)$$

The integrability condition for this partial differential equation reads

$$0 = \frac{1}{2} t^2 (d\nu_s + \nu_f \wedge \nu_s) + t (d\nu_f + \hat{\nu}_s \wedge \nu_s) + (d\hat{\nu}_s + \hat{\nu}_s \wedge \nu_f) \quad (15)$$

which, for fixed $p \in M^{n-1}$, is a quadratic polynomial in t :

Theorem. *If there are more than two hypersurfaces orthogonal to all circles of a congruence $c : M^{n-1} \rightarrow G_+(n-1, 3)$, then the circle congruence is normal.*

Obviously, this statement is due to the fact that a quadratic polynomial must vanish identically if it has more than two zeros. Now, the coefficients in (15) are exactly the coefficients in the curvature form $d\nu + \nu \wedge \nu$ of the vector bundle c^\perp over M^{n-1} , i.e.

Theorem. *A circle congruence $c : M^{n-1} \rightarrow G_+(n-1, 3)$ is normal if and only if the vector bundle c^\perp over M^{n-1} is flat.*

The differential equation (14) for t becomes trivial exactly when the basis fields s , f and \hat{f} of c^\perp are parallel — then, t can be considered a “real” parameter for the 1-parameter family $(f_t)_t$ of orthogonal hypersurfaces of the cyclic system. Recalling the geometric interpretation (2) of $g(t)$ in the parametrization (12) we obtain

Theorem. *Any four orthogonal hypersurfaces of a cyclic system intersect all circles at a fixed cross ratio.*

And finally, since $s : M^{n-1} \rightarrow S_1^{n+1}$ is a Ribaucour sphere congruence, we come back to our starting point:

Theorem. *Any two orthogonal hypersurfaces of a cyclic system envelope a Ribaucour sphere congruence. The circles that intersect the spheres of a Ribaucour congruence orthogonally in its two envelopes form a cyclic system.*

7. Triply orthogonal systems

We just learned that any two hypersurfaces orthogonal to the circles of a cyclic system envelope a Ribaucour sphere congruence. Consequently, the curvature directions on all orthogonal hypersurfaces of a cyclic system do correspond: integrating the $(n-1)$ curvature directions on one orthogonal hypersurface we obtain $(n-1)$ 1-parameter families of 2-dimensional surfaces — each surface built up from the circles along one curvature line — which intersect all orthogonal hypersurfaces in their curvature lines. In case of 3-dimensional ambient space¹⁴⁾ this yields what is called a “triply orthogonal system” of surfaces:

Definition (triply orthogonal system). *A system of three 1-parameter families of surfaces in a 3-dimensional space Q_k^3 is called a triply orthogonal system if any two surfaces from different families intersect orthogonally.*

Classically, triply orthogonal systems were considered in Euclidean ambient space (see for example [8]) but the notion of a triply orthogonal system is obviously conformally invariant: a generalized stereographic projections $Q_k^3 \rightarrow Q_{\tilde{k}}^3$

¹⁴⁾ Note, that the situation is rather special in 3-dimensional ambient space: in higher dimensions the curvature lines of a hypersurface generally do not come from a coordinate system.

will map any triply orthogonal system in Q_k^3 onto one in Q_k^3 — and so will any Möbius transformation do. Since the curvature directions of a surface in Q_k^3 are invariant under the stereographic projections, too, Dupin's theorem [8] on triply orthogonal systems in Euclidean 3-space holds in spaces of constant curvature:

Theorem (Dupin). *The surfaces of a triply orthogonal system in Q_k^3 intersect along their curvature lines.*

Applying this theorem to a triply orthogonal system coming from a cyclic system in Q_k^3 , we conclude that the two families of surfaces orthogonal to the orthogonal surfaces of the cyclic system consist of channel surfaces: these surfaces carry one family of circular curvature lines and, consequently¹⁵⁾, each surface envelopes a 1-parameter family of spheres (compare [2]). In fact, this is the characterization of cyclic systems from the viewpoint of triply orthogonal systems:

Theorem. *A triply orthogonal system comes from a cyclic system if and only if two of the 1-parameter families of surfaces consist of channel surfaces.*

Before discussing Guichard's nets in the second part of this paper it remains to learn some facts about triply orthogonal systems in general: given a triply orthogonal system in parametric form $(t_1, t_2, t_3) \mapsto f(t_1, t_2, t_3) \in Q_k^3$, i.e. the surfaces of the system being given by $t_i = \text{const}$, we may choose a pseudo orthonormal framing $F = (n_1, n_2, n_3, f, \hat{f}) : M^3 \rightarrow O_1(5)$ wherein n_i denote the unit normal fields — or, according to our previous identification of the Lorentz sphere with the space of (oriented) hyperspheres in Q_k^3 , the tangent planes — of the surfaces $t_i = \text{const}$ and $\hat{f} = n_k + \frac{k}{2}f$ describes the second intersection point of the n_i . The connection form $\Phi = F^{-1}dF$ of such a framing is of the form

$$\Phi = \begin{pmatrix} 0 & -k_{21}\omega_1 + k_{12}\omega_2 & -k_{31}\omega_1 + k_{13}\omega_3 & \omega_1 & \frac{k}{2}\omega_1 \\ k_{21}\omega_1 - k_{12}\omega_2 & 0 & -k_{32}\omega_2 + k_{23}\omega_3 & \omega_2 & \frac{k}{2}\omega_2 \\ k_{31}\omega_1 - k_{13}\omega_3 & k_{32}\omega_2 - k_{23}\omega_3 & 0 & \omega_3 & \frac{k}{2}\omega_3 \\ -\frac{k}{2}\omega_1 & -\frac{k}{2}\omega_2 & -\frac{k}{2}\omega_3 & 0 & 0 \\ -\omega_1 & -\omega_2 & -\omega_3 & 0 & 0 \end{pmatrix} \quad (16)$$

where $\omega_i = l_i dt_i$ with Lamé's functions $l_i := |\frac{\partial f}{\partial t_i}|$ and $k_{ij} = -\frac{1}{l_i l_j} \frac{\partial}{\partial t_i} l_j$ give the principal curvatures of the surfaces $t_i = \text{const}$ in t_j -direction. The Maurer-Cartan equation $d\Phi + \Phi \wedge \Phi = 0$ reduces to Lamé's equations

$$\begin{aligned} k &= e_1 k_{12} + e_2 k_{21} - k_{12}^2 - k_{21}^2 - k_{31} k_{32} \\ k &= e_2 k_{23} + e_3 k_{32} - k_{23}^2 - k_{32}^2 - k_{12} k_{13} \\ k &= e_3 k_{31} + e_1 k_{13} - k_{31}^2 - k_{13}^2 - k_{23} k_{21} \end{aligned} \quad (17)$$

$$\begin{aligned} 0 &= e_1 k_{23} + k_{13}(k_{21} - k_{23}) & 0 &= e_1 k_{32} + k_{12}(k_{31} - k_{32}) \\ 0 &= e_2 k_{31} + k_{21}(k_{32} - k_{31}) & 0 &= e_2 k_{13} + k_{23}(k_{12} - k_{13}) \\ 0 &= e_3 k_{12} + k_{32}(k_{13} - k_{12}) & 0 &= e_3 k_{21} + k_{31}(k_{23} - k_{21}) \end{aligned}$$

where $e_i := \frac{1}{l_i} \frac{\partial}{\partial t_i}$ are the unit vector fields in t_i -direction. In fact, as the Maurer-Cartan equation for the connection form (16) Lamé's equations are exactly the conditions on three functions l_i , $i = 1, 2, 3$, to determine a triply orthogonal system:

¹⁵⁾ This is a consequence of Joachimsthal's theorem.

Theorem (Lamé). *Three functions l_1 , l_2 and l_3 are the Lamé functions of a triply orthogonal system in a quadric Q_k^3 of constant curvature k if and only if they satisfy Lamé's equations (17) with $e_i = \frac{1}{l_i} \frac{\partial}{\partial t_i}$ and $k_i = -\frac{1}{l_i l_j} \frac{\partial}{\partial t_i} l_j$.*

In some situations — especially when examining Guichard's nets — it is more convenient to allow the parametrization f *not* to take values in one quadric Q_k^3 but, more generally, in the light cone L^{n+1} . Then, the unit vectors $n_i // \frac{\partial f}{\partial t_i}$ will not longer describe the tangent plane congruences of the surfaces $t_i = \text{const}$ but sphere congruences which are enveloped by the surfaces. On the connection form (16) this has the effect that $d\hat{f} = \sum_{i,j=1}^3 b_{ij} n_i \omega_j$ where the functions b_{ij} can be determined from the Maurer-Cartan equation:

$$\begin{aligned} b_{11} + b_{22} &= e_1 k_{12} + e_2 k_{21} - k_{12}^2 - k_{21}^2 - k_{31} k_{32} \\ b_{22} + b_{33} &= e_2 k_{23} + e_3 k_{32} - k_{23}^2 - k_{32}^2 - k_{12} k_{13} \\ b_{33} + b_{11} &= e_3 k_{31} + e_1 k_{13} - k_{31}^2 - k_{13}^2 - k_{23} k_{21} \\ b_{21} &= e_1 k_{23} + k_{13}(k_{21} - k_{23}) & b_{31} &= e_1 k_{32} + k_{12}(k_{31} - k_{32}) \\ b_{32} &= e_2 k_{31} + k_{21}(k_{32} - k_{31}) & b_{12} &= e_2 k_{13} + k_{23}(k_{12} - k_{13}) \\ b_{13} &= e_3 k_{12} + k_{32}(k_{13} - k_{12}) & b_{23} &= e_3 k_{21} + k_{31}(k_{23} - k_{21}) \end{aligned} \quad (18)$$

with e_i and k_{ij} as above. Note, that $b_{ij} = b_{ji}$. The integrability conditions for Lamé's functions now become third order differential equations — the conformal flatness¹⁶⁾ of the metric $\langle df, df \rangle = \sum_{i=1}^3 l_i^2 dt_i^2$:

$$\begin{aligned} e_1 b_{23} - k_{23} b_{31} - k_{13} b_{23} &= e_3 b_{21} - k_{21} b_{13} - k_{31} b_{21} \\ e_2 b_{31} - k_{31} b_{12} - k_{21} b_{31} &= e_1 b_{32} - k_{32} b_{21} - k_{12} b_{32} \\ e_3 b_{12} - k_{13} b_{23} - k_{32} b_{12} &= e_2 b_{13} - k_{13} b_{32} - k_{23} b_{13} \\ 0 &= e_1 b_{22} - e_2 b_{21} + k_{12}(b_{11} - b_{22}) + k_{21}(b_{12} + b_{21}) + k_{32} b_{31} \\ 0 &= e_2 b_{33} - e_3 b_{32} + k_{23}(b_{22} - b_{33}) + k_{32}(b_{23} + b_{32}) + k_{13} b_{12} \\ 0 &= e_3 b_{11} - e_1 b_{13} + k_{31}(b_{33} - b_{11}) + k_{13}(b_{31} + b_{13}) + k_{21} b_{23} \\ 0 &= e_2 b_{11} - e_1 b_{12} + k_{21}(b_{22} - b_{11}) + k_{12}(b_{21} + b_{12}) + k_{31} b_{32} \\ 0 &= e_3 b_{22} - e_2 b_{23} + k_{32}(b_{33} - b_{22}) + k_{23}(b_{32} + b_{23}) + k_{12} b_{13} \\ 0 &= e_1 b_{33} - e_3 b_{31} + k_{13}(b_{11} - b_{33}) + k_{31}(b_{13} + b_{31}) + k_{23} b_{21} \end{aligned} \quad (19)$$

where $e_i = \frac{1}{l_i} \frac{\partial}{\partial t_i}$, $k_{ij} = -\frac{1}{l_i l_j} \frac{\partial}{\partial t_i} l_j$ and b_{ij} are defined by (18). These are the equations we will use later — in place of the original Lamé equations (17) — when discussing Guichard's nets in Q_k^3 .

¹⁶⁾ In fact, Lamé's equations (17) are the conditions on the metric $\sum_{i=1}^3 l_i^2 dt_i^2$ to have constant curvature k .

Part II: Guichard's nets

In the second part we are going to classify those Guichard nets — special triply orthogonal systems — which come from cyclic systems. A Guichard net can be considered as a 3-dimensional analog of an isothermic net in the plane [3]— the surfaces of a Guichard net divide the ambient space into infinitesimal rectangular parallelepipeds (any two surfaces of different families intersect orthogonally along curvature lines) such that two of the six diagonal rectangles are squares:

Definition (Guichard net). *A triply orthogonal system is called a Guichard net if, with a suitable choice of $\varepsilon_i \in \{1, i\}$, its Lamé functions l_i satisfy*

$$\sum_{i=1}^3 (\varepsilon_i l_i)^2 = 0. \quad (20)$$

As we will see, a cyclic system which gives rise to a Guichard net consists of a family of parallel Weingarten surfaces in some space of constant curvature¹⁷⁾. Thus, as a byproduct, we obtain several generalizations of Bonnet's theorem on parallel surfaces of constant mean curvature in Euclidean space [7]. To complete the discussion, we will try to give an “estimate” for the generality of those Guichard nets coming from cyclic systems by discussing the effect of the assumption to come from a cyclic system on Lamé's equations (19).

1. Cyclic systems

Let us start with a cyclic system $c : M^2 \rightarrow G_+(2, 3)$ — we assume that the family of its orthogonal surfaces is given by $f_t : M^2 \rightarrow L^4$ such that t is the arc length on all circles simultaneously, i.e. $|\frac{\partial}{\partial t} f_t| \equiv 1$ which can always be achieved by a suitable scaling of f_t into the light cone. Denoting $f = f_0$ we find that

$$f_t = \frac{1}{g'(t)} [g(t) s + f - \frac{1}{2} g^2(t) \hat{f}] \quad (21)$$

with a Ribaucour sphere congruence s , its second envelope \hat{f} and where g denotes some function with $g(0) = 0$ and $g'(0) = 1$ (see (12)). The condition (20) for the corresponding triply orthogonal system to be a Guichard net reads — since all circles are simultaneously parametrized by arc length —

$$\left| \frac{\partial}{\partial t_1} f_t \right|^2 + \varepsilon^2 \left| \frac{\partial}{\partial t_2} f_t \right|^2 = 1$$

where (t_1, t_2) are suitable curvature line coordinates and $\varepsilon \in \{1, i\}$, depending on the position of the diagonal infinitesimal squares in the Guichard net relativ

¹⁷⁾ This behaviour is similar to that of isothermic Willmore surfaces (Thomsen's theorem [1]): a surface belonging to two Möbius geometric surface classes turns out to belong to a metric surface class — minimal surfaces in spaces of constant curvature.

to the circle direction. The Guichard condition for $t = 0$ gives us

$$df = \begin{cases} \cos u \cdot s_1 dt_1 + \sin u \cdot s_2 dt_2 \\ \cosh u \cdot s_1 dt_1 + \sinh u \cdot s_2 dt_2 \end{cases}$$

with $s_1, s_2 : M^2 \rightarrow S_1^4$ and a suitable function $u : M^2 \rightarrow \mathbb{R}$. With the ansatz

$$\begin{aligned} ds &= \begin{cases} -(a_1 \cos u - a_2 \sin u)s_1 dt_1 - (a_1 \sin u + a_2 \cos u)s_2 dt_2 \\ -(a_1 \cosh u + a_2 \sinh u)s_1 dt_1 - (a_1 \sinh u + a_2 \cosh u)s_2 dt_2 \end{cases} \\ d\hat{f} &= \begin{cases} (b_1 \cos u - b_2 \sin u)s_1 dt_1 + (b_1 \sin u + b_2 \cos u)s_2 dt_2 \\ (b_1 \cosh u + b_2 \sinh u)s_1 dt_1 + (b_1 \sinh u + b_2 \cosh u)s_2 dt_2 \end{cases} \end{aligned} \quad (22)$$

where $a_i, b_i : M^2 \rightarrow \mathbb{R}$ denote suitable functions the Guichard condition becomes

$$\begin{aligned} g'^2 &= (1 - a_1 g - \tfrac{1}{2}b_1 g^2)^2 + \varepsilon^2 (a_2 g + \tfrac{1}{2}b_2 g^2)^2 \\ &= [1 - (a_1 + \varepsilon a_2)g - \tfrac{1}{2}(b_1 + \varepsilon b_2)g^2] \cdot [1 - (a_1 - \varepsilon a_2)g - \tfrac{1}{2}(b_1 - \varepsilon b_2)g^2]. \end{aligned}$$

This equation has some interesting consequences: first, the function g has to be an elliptic function. Since it does not depend on (t_1, t_2) its branch points do not either. Hence, a_i and b_i are constant and, consequently, there is a constant vector $n_k := b_2 s + (a_1 b_2 - a_2 b_1)f + a_2 \hat{f}$ perpendicular to all s_1 and s_2 :

$$dn_k = d[b_2 s + (a_1 b_2 - a_2 b_1)f + a_2 \hat{f}] = 0.$$

Without loss of generality¹⁸⁾, we may assume $n_k \neq 0$: then, the cyclic system $c = \text{span}\{s_1, s_2\} : M^2 \rightarrow G_+(2, 3)$ consists of straight lines in the quadric Q_k^3 of constant sectional curvature $k = -\langle n_k, n_k \rangle$ corresponding to the vector n_k , and the surfaces f_t are parallel in Q_k^3 .

Proposition. *Any cyclic system which defines a Guichard net is a normal line congruence in some quadric Q_k^3 of constant curvature.*

Finally, we find that the Maurer-Cartan equation for the adapted principal curvature framing $F = (s_1, s_2, s, f, \hat{f}) : M^2 \rightarrow O_1(5)$ reduces to some version of the sine-Gordon equation:

$$\begin{aligned} 0 &= \left(\frac{\partial}{\partial t_1} \frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2} \frac{\partial}{\partial t_2} \right) u + \frac{1}{2}([a_1^2 - a_2^2 + 2b_1] \sin 2u + [a_1 a_2 + b_2] \cos 2u), \\ 0 &= \left(\frac{\partial}{\partial t_1} \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \frac{\partial}{\partial t_2} \right) u + \frac{1}{2}([a_1^2 + a_2^2 + 2b_1] \sinh 2u + [a_1 a_2 + b_2] \cosh 2u) \end{aligned}$$

depending on whether $\varepsilon = 1$ or $\varepsilon = i$, respectively.

¹⁸⁾ The vector n_k vanishes if and only if $a_2 = b_2 = 0$. But in this case all surfaces f_t are totally umbilic — they form a sphere pencil. The corresponding conformally flat hypersurfaces were completely classified in [5].

2. Parallel Weingarten surfaces

Now, that we know that our cyclic system is in fact a normal line congruence in some space of constant curvature we may choose a more adapted parametrization for the family of orthogonal surfaces: after choosing a “base surface” $f = f_0$ out of the family of parallel surfaces — and assuming f to actually take values in Q_k^3 , i.e. $\langle f, n_k \rangle = 1$ — we might fix the Ribaucour sphere congruence s to be its tangent plane congruence, i.e. $\langle s, n_k \rangle = 0$. For the vector n_k defining the quadric Q_k^3 this means

$$n_k = -\frac{k}{2}f + \hat{f}$$

and the second envelope \hat{f} of s becomes trivial: it is the antipode surface of f , the point at infinity or the reflection of f at the infinity boundary depending on whether $k > 0$, $k = 0$ or $k < 0$, respectively. In (22), this is reflected by the fact that $2b_1 = k$ and $b_2 = 0$. This way, we have fixed the frame (s, f, \hat{f}) which we use to parametrize the circles of the congruence.

Instead of using arc length parametrizations (3) — which would lead to unpleasant calculations — we choose the easiest parametrization possible for the tangent plane congruences s_t of the parallel surfaces f_t :

$$s_t := \frac{1}{\sqrt{1+kt^2}} \left(s + t \cdot \left[\frac{k}{2}f + \hat{f} \right] \right).$$

Asking now all $f_t : M^2 \rightarrow Q_k^3$ to take values in Q_k^3 , i.e. fixing the scaling of f_t “correctly”, gives us

$$\begin{aligned} f_t &= \frac{1}{\sqrt{1+kt^2}} \left(f - t \cdot \left[s + \frac{t}{1+\sqrt{1+kt^2}} \cdot n_k \right] \right), \\ \hat{f}_t &= \frac{1}{\sqrt{1+kt^2}} \left(\hat{f} - \frac{kt}{2} \cdot \left[s + \frac{t}{1+\sqrt{1+kt^2}} \cdot n_k \right] \right). \end{aligned} \quad (23)$$

Herein, the range where t is running is restricted by the condition $1+kt^2 > 0$: in case $k < 0$ this prevents us from running through the infinity boundary — as $t \rightarrow \pm \frac{1}{\sqrt{-k}}$ the points of f_t approach the intersection points of each circle with the infinity sphere n_k , as the points of \hat{f}_t do “from the other side”. In case of Euclidean ambient space, $k = 0$, f_t parametrizes each circle up to the point $n_0 \equiv \hat{f}_t$ at infinity which is approached as $t \rightarrow \pm\infty$. Finally, in case $k > 0$, f_t also parametrizes just half of each great circle, \hat{f}_t taking the other half; the two antipode points $f_{\pm\infty} = \mp \frac{1}{\sqrt{k}}s + \frac{1}{k}n_k$ which lie symmetric with respect to f_0 and \hat{f}_0 are never reached but just approached by f_t and \hat{f}_t as $t \rightarrow \pm\infty$.

Since s_t is the tangent plane congruence of f_t , i.e. it can be interpreted as the normal field of $f_t : M^2 \rightarrow Q_k^3$, we can easily calculate the first and second fundamental forms of f_t — there is no need to calculate those of \hat{f}_t since \hat{f}_t is *isometric* to f_t in Q_k^3 :

$$I_t = \langle df_t, df_t \rangle \quad \text{and} \quad II_t = -\langle ds_t, df_t \rangle.$$

With (22) — remember that we have $b_1 = \frac{k}{2}$ and $b_2 = 0$ since s is the tangent plane congruence of $f = f_0$ in Q_k^3 — we find

$$k_1 = \begin{cases} \frac{(a_1 - kt) \cos u - a_2 \sin u}{(1 + a_1 t) \cos u - a_2 t \sin u} \\ \frac{(a_1 - kt) \cosh u + a_2 \sinh u}{(1 + a_1 t) \cosh u + a_2 t \sinh u} \end{cases} \quad \text{and} \quad k_2 = \begin{cases} \frac{(a_1 - kt) \sin u + a_2 \cos u}{(1 + a_1 t) \sin u + a_2 t \cos u} \\ \frac{(a_1 - kt) \sinh u + a_2 \cosh u}{(1 + a_1 t) \sinh u + a_2 t \cosh u} \end{cases}$$

for the principal curvatures of f_t . Since in both cases, $\varepsilon = 1$ and $\varepsilon = i$, both principal curvatures are given in terms of *one* function, u , the surfaces f_t clearly are Weingarten. Moreover, it is easy to see that the (extrinsic) Gauß curvature $k_t = k_1 k_2$ and the mean curvature $H_t = \frac{1}{2}(k_1 + k_2)$ of the surfaces $f_t : M^2 \rightarrow Q_k^3$ satisfy an affine relation $0 = c_K(t) \cdot K_t + 2c_H(t) \cdot H_t + c(t)$ where

$$\begin{aligned} c_K(t) &= [a_1^2 + (\varepsilon a_2)^2] \cdot t^2 + 2a_1 \cdot t + 1, \\ c_H(t) &= a_1 k \cdot t^2 + [k - (a_1^2 + (\varepsilon a_2)^2)] \cdot t - a_1, \\ c(t) &= k^2 \cdot t^2 - 2a_1 k \cdot t + [a_1^2 + (\varepsilon a_2)^2]. \end{aligned} \quad (24)$$

Note, that the sign of

$$c_K(t) \cdot c(t) - c_H^2(t) = \varepsilon^2 a_2^2 (1 + kt^2)^2 \quad (25)$$

is an invariant of the family of parallel Weingarten surfaces: it determines the position of the infinitesimal squares in the Guichard net relative to the family's normal direction.

Proposition. *The orthogonal surfaces of a cyclic system defining a Guichard net are (parallel) linear Weingarten surfaces in a quadric Q_k^3 of constant curvature, i.e. their (extrinsic) Gauß and mean curvature satisfy an affine relation*

$$c_K(t) \cdot K_t + 2c_H(t) \cdot H_t + c(t) = 0.$$

Since we lately changed the parametrization (23) of the cyclic system we need to derive the condition on the triply orthogonal net to form a Guichard net again: to that extend suppose $t = t(r)$ is a function of a new parameter r . Then, the condition $|\frac{\partial f_t}{\partial r}|^2 = |\frac{\partial f_t}{\partial t_1}|^2 + \varepsilon^2 |\frac{\partial f_t}{\partial t_2}|^2$ to form a Guichard net becomes

$$t'^2 = [1 + kt^2] \cdot c_K(t) \quad (26)$$

which, again, is the equation of an elliptic function. From this equation we first see that in case $k < 0$ of hyperbolic ambient space Guichard nets do *not* extend through the infinity sphere: for $t \rightarrow \pm \frac{1}{\sqrt{-k}}$ the Guichard net becomes singular. Thus, any Guichard net defined through a cyclic system is a normal line congruence in a *space form* rather than just in one of the constant curvature quadrics Q_k^3 . Also, any cyclic Guichard net becomes singular at the zeros of c_K : these are surfaces of constant mean curvature which might occur in the family of parallel Weingarten surfaces.

Proposition. *A Guichard net given by a cyclic system becomes singular at any constant mean curvature surface present in the family of orthogonal surfaces of the cyclic system as well as it becomes singular at the infinity sphere in case of hyperbolic ambient space.*

Examining the occurrence of “special” surfaces — characterized through the zeros of c_K , c_H or c — in a family of parallel Weingarten surfaces is interesting on its own; but, at the moment, we would like to postpone this topic and rather discuss the

3. Construction of Guichard nets

which come from cyclic systems: in the preceeding two paragraphs we learned that those Guichard nets always come from a family of parallel Weingarten surfaces whose Gauß and mean curvature satisfy an affine relation. To understand the reverse, we start with a Weingarten surface $f : M^2 \rightarrow Q_k^3$ whose principal curvatures k_1 and k_2 satisfy $0 = c_K k_1 k_2 + c_H(k_1 + k_2) + c$ and study the triply orthogonal system given by the family of its parallel surfaces: since surfaces of constant mean curvature are singular for cyclic Guichard nets we exclude them — hence, we may assume $c_K = 1$. Now, the principal curvature coordinates of f can be fixed in a canonical way: setting $a_1 := -c_H$ and $a_2 := \sqrt{|c - c_H^2|}$ and u denoting a suitable function, we make the ansatz

$$k_1 = \begin{cases} a_1 - a_2 \tan u \\ a_1 + a_2 \tanh u \end{cases} \quad \text{and} \quad k_2 = \begin{cases} a_1 + a_2 \cot u \\ a_1 + a_2 \coth u \end{cases} \quad (27)$$

in case $c - c_H^2 > 0$ and in case $c - c_H^2 < 0$, respectively — at this point we exclude $c - c_H^2 = 0$: surfaces with $0 = (k_1 + c_H)(k_2 + c_H)$, i.e. which have one constant principal curvature. Besides for sphere pieces, surfaces of this kind do *not* occur as orthogonal surfaces of a cyclic system which defines a Guichard net (compare (25)). With the above ansatz for the principal curvatures the Codazzi equations for f read in principal curvature coordinates (t_1, t_2)

$$\begin{aligned} \frac{\partial}{\partial t_2} \left(\frac{\sqrt{\langle \frac{\partial}{\partial t_1} f, \frac{\partial}{\partial t_1} f \rangle}}{\cos u} \right) &= 0 & \frac{\partial}{\partial t_1} \left(\frac{\sqrt{\langle \frac{\partial}{\partial t_2} f, \frac{\partial}{\partial t_2} f \rangle}}{\sin u} \right) &= 0 \\ \frac{\partial}{\partial t_2} \left(\frac{\sqrt{\langle \frac{\partial}{\partial t_1} f, \frac{\partial}{\partial t_1} f \rangle}}{\cosh u} \right) &= 0 & \frac{\partial}{\partial t_1} \left(\frac{\sqrt{\langle \frac{\partial}{\partial t_2} f, \frac{\partial}{\partial t_2} f \rangle}}{\sinh u} \right) &= 0 \end{aligned} \quad \text{and}$$

showing that with $\varepsilon \in \{1, i\}$ — $\varepsilon = 1$ if $c - c_H^2 > 0$ and $\varepsilon = i$ if $c - c_H^2 < 0$ — we can assume

$$\left| \frac{\partial}{\partial t_1} f \right|^2 + \varepsilon^2 \left| \frac{\partial}{\partial t_2} f \right|^2 \equiv \text{const.}$$

Now, parametrizing the family of parallel surfaces of our Weingarten surface — as for example in (23) with the unit normal field $s = n : M^2 \rightarrow S_1^4$ of f — we see that running through it with the “correct speed” (26) provides us with a Guichard net.

Theorem (cyclic Guichard nets). *The orthogonal surfaces of any cyclic system which defines a Guichard net are parallel linear Weingarten surfaces in a space form, i.e. their Gauß and mean curvatures satisfy an affine relation*

$$c_K K + 2c_H H + c = 0$$

where $c_K \neq 0$ as well as $c_K c - c_H^2 \neq 0$ (no surfaces of constant mean curvature or with a constant principal curvature occur¹⁹). Conversely, the normal line congruence of such a Weingarten surface always defines a cyclic Guichard net.

To estimate the “amount of generality” of those

¹⁹⁾ Besides the degenerate case of sphere pieces: those Guichard nets were discussed in [5].

4. Guichard's nets

coming from cyclic systems we will study the effect of the assumption to come from a cyclic system Lamé's equations (19) for any Guichard net: choosing an appropriate scaling for the parametrization $f : M^3 \rightarrow L^4$ of a Guichard net in a quadric Q_k^3 of constant curvature we may assume $l_3 \equiv 1$ for one of its Lamé functions. Then, the condition to be Guichard reads $l_1^2 + \varepsilon^2 l_2^2 = 1$ which leads to the ansatz $l_1 = \cos(\varepsilon w)$ and $l_2 = \frac{1}{\varepsilon} \sin(\varepsilon w)$ where $w : M^3 \rightarrow \mathbb{R}$ denotes a suitable function. Introducing functions

$$\begin{aligned} w_0 &:= \left(\frac{\partial w}{\partial t_1}\right)^2 + \varepsilon^2 \left(\frac{\partial w}{\partial t_2}\right)^2 + \left(\frac{\partial w}{\partial t_3}\right)^2 \\ w_1 &:= -\varepsilon \frac{\partial}{\partial t_1} \frac{\partial w}{\partial t_3} \cdot \cot(\varepsilon w) \\ w_2 &:= \varepsilon \frac{\partial}{\partial t_2} \frac{\partial w}{\partial t_3} \cdot \tan(\varepsilon w) \\ w_3 &:= \frac{\varepsilon}{\sin(2\varepsilon w)} \left(\left(\frac{\partial}{\partial t_1} \frac{\partial}{\partial t_1} - \varepsilon^2 \frac{\partial}{\partial t_2} \frac{\partial}{\partial t_2} \right) w - \frac{\partial}{\partial t_3} \frac{\partial w}{\partial t_3} \cdot \cos(2\varepsilon w) \right) \end{aligned}$$

the generalized Lamé's equations reduce to four differential equations

$$\frac{\partial w_i}{\partial t_j} = \frac{\partial w_j}{\partial t_i} \quad \text{and} \quad \frac{\partial w_1}{\partial t_1} + \varepsilon^2 \frac{\partial w_2}{\partial t_2} + \frac{\partial w_3}{\partial t_3} = \varepsilon^2 \frac{\partial w_0}{\partial t_3}, \quad (28)$$

i.e. (w_1, w_2, w_3) is a gradient and $\text{div}(w_1, w_2, w_3) = \frac{\partial}{\partial t_3} |\text{grad} w|^2$. Comparing the present ansatz to our first ansatz (21) for a Guichard net coming from a cyclic system and using (22) we see that in case the Guichard net comes from a cyclic system we have

$$\begin{aligned} \cos(\varepsilon w) &= \frac{1}{g'} ([1 - a_1 g - \frac{1}{2} b_1 g^2] \cos(\varepsilon u) + [a_2 g + \frac{1}{2} b_2 g^2] \varepsilon \sin(\varepsilon u)) \\ \sin(\varepsilon w) &= \frac{\varepsilon}{g'} ([1 - a_1 g - \frac{1}{2} b_1 g^2] \frac{1}{\varepsilon} \sin(\varepsilon u) - [a_2 g + \frac{1}{2} b_2 g^2] \cos(\varepsilon u)) \end{aligned}$$

and, consequently, the function w has to split: $w(t_1, t_2, t_3) = u(t_1, t_2) + v(t_3)$. On Lamé's equations (28) this has a radical effect: it remains only *one* single equation

$$\frac{\varepsilon}{\sin(2\varepsilon w)} \left(\left(\frac{\partial}{\partial t_1} \frac{\partial}{\partial t_1} - \varepsilon^2 \frac{\partial}{\partial t_2} \frac{\partial}{\partial t_2} \right) u - v'' \cos(2\varepsilon w) \right) - \varepsilon^2 (v')^2 = c = \text{const}$$

which is equivalent to

$$\begin{aligned} \left(\frac{\partial}{\partial t_1} \frac{\partial}{\partial t_1} - \varepsilon^2 \frac{\partial}{\partial t_2} \frac{\partial}{\partial t_2} \right) u &= [(c + \varepsilon^2 v'^2) \cos(2\varepsilon v) - v'' \varepsilon \sin(2\varepsilon v)] \cdot \frac{1}{\varepsilon} \sin(2\varepsilon u) \\ &+ [(c + \varepsilon^2 v'^2) \frac{1}{\varepsilon} \sin(2\varepsilon v) - v'' \cos(2\varepsilon v)] \cdot \cos(2\varepsilon u). \end{aligned}$$

Herein, the coefficients of $\sin(2\varepsilon u)$ and $\cos(2\varepsilon u)$ have to be constant which "splits" the equation: for the function $u = u(t_1, t_2)$ we obtain a version of the sine-Gordon (sinh-Gordon) equation and for the function $v = v(t_3)$ we get a modified pendulum equation

$$c + \varepsilon^2 v'^2 = r_0 \cos(2\varepsilon[v - v_0]).$$

Thus, as in our previous ansatzes, the circles of a cyclic Guichard net are simultaneously parametrized by an elliptic function v . These discussions may convince

the reader that those Guichard nets in a quadric Q_k^3 coming from cyclic systems are quite special — even though we are far from being able to solve the system (28) it seems reasonable to expect the existence of more general solutions w that do not split into a function u of two and one, v , of one variable only. . .

Because of the relation between systems of parallel Weingarten surfaces in space forms and cyclic Guichard nets — which provides us with a Möbius geometric characterization for those systems of parallel Weingarten surfaces — we may use the Möbius geometric setting to study linear Weingarten surfaces. To present this principle we will give simple proofs for various generalizations of

5. Bonnet's theorem

on the existence of a parallel constant mean curvature surface to a given surface of constant mean curvature [7]. From our main theorem on cyclic Guichard nets (p.21) we know that the family of parallel surfaces of a linear Weingarten surface — which has non constant principal curvatures — in a space form of sectional curvature k are linear Weingarten surfaces, i.e. the (extrinsic) Gauß and mean curvature of all surfaces f_t in the family satisfy an affine relation

$$0 = c_K(t) \cdot K_t + 2c_H(t) \cdot H_t + c(t).$$

In this setting, surfaces of constant Gauß curvature, of constant mean curvature or with constant sum $\frac{1}{k_1} + \frac{1}{k_2}$ of the curvature radii are characterized by the zeros of c_H , c_K and c , respectively. If we parametrize the family f_t as in (23) then c_K , c_H and c become the quadratic polynomials (24) with $a_1 = -c_H(0)$ and $a_2 = \sqrt{|c(0) - c_H^2(0)|}$ — here, we assume that f_0 is *not* a surface of constant mean curvature²⁰⁾ so that the ansatz (27) for the principal curvatures works and that we can assume $c_K(0) = 1$, without loss of generality. Thus, generalizations of Bonnet's theorem can be obtained by studying the zeros of quadratic polynomials (compare [7]) — the only thing which remains unpleasant is the calculation of the distances between two surfaces of the family: to that purpose we have to integrate

$$\int \left| \frac{\partial}{\partial t} f_t \right| dt = \int \frac{dt}{1 + kt^2}.$$

Let us summarize these observations in a

Theorem (meta theorem). *In the Möbius geometric setting, the study of parallel surfaces of constant Gauß and mean curvature and with constant sum of the curvature radii reduces to the study of zeros of quadratic polynomials.*

In case $\varepsilon = 1$, i.e. $c_K c - c_H^2 > 0$, neither c_K nor c have real zeros showing that there are no constant mean curvature surfaces or surfaces with constant sum of their curvature radii in the family. c_H , on the other hand, has always

²⁰⁾ This is no restriction: to a surface of constant mean curvature in any space form there exist plenty of parallel surfaces which do not have constant mean curvature.

zeros — in case $k < 0$, one of them lies in $(-\frac{1}{\sqrt{-k}}, \frac{1}{\sqrt{-k}})$. Consequently, there is always (at least) one surface of constant Gauß curvature present which we can choose as the “base surface” f_0 of the family. But $c_H(0) = 0$ means that c_H is, in fact, linear and hence has exactly *one* zero — provided it does not vanish identically. In case of Euclidean and hyperbolic ambient spaces, this means that there is exactly *one* surface of constant Gauß curvature in the family. In case of elliptic ambient space, it is easy to see that the surfaces $f_{\pm\infty}$ have constant Gauß curvature

$$K_{\pm\infty} = \lim_{t \rightarrow \infty} \frac{-1}{c_K(t)}(c(t) + 2c_H(t)H_t) = -\frac{k^2}{a_2^2},$$

too. Thus, in case of elliptic ambient space there exist *four* surfaces of constant Gauß curvature in the family which have distances $\frac{1}{2\sqrt{k}}\pi$, i.e. they divide each normal great circle in quarters. Since the antipode surface of a surface clearly has the same curvatures as the surface itself there occur two values K_0 and K_∞ for the constant Gauß curvatures — these satisfy $K_0 K_\infty = k^2$.

In case $\varepsilon = i$, i.e. $c_K c - c_H^2 < 0$, the situation becomes more interesting: if $k > 0$, i.e. in case of elliptic ambient space, all of the functions c_K , c_H and c have real zeros: as for $\varepsilon = 1$, there are two parallel surfaces of constant Gauß curvatures K_1 and K_2 , $K_1 K_2 = k^2$, in distance $\frac{\pi}{2\sqrt{k}}$ — and their antipode surfaces — there are two (antipode) pairs of parallel surfaces of constant mean curvature $\pm \frac{1}{2}(\sqrt{K_1} - \sqrt{K_2})$ in distance $d = \frac{1}{\sqrt{k}} \arctan \sqrt{\frac{k}{K_1}}$ from the K_1 -surfaces and two pairs of constant $\pm \frac{1}{2k}(\sqrt{K_1} - \sqrt{K_2})$ sum of their curvature radii, in the same distance d from the K_2 -surfaces.

If $k = 0$, i.e. in case of Euclidean ambient space, the function c_H becomes linear, c_K has two real zeros and c has no zeros: we obtain Bonnet’s classical theorem — provided c_H does not vanish identically in which case we are left with a family of surfaces of constant sum of their curvature radii, parallel to a minimal surface.

If $k < 0$, i.e. in case of hyperbolic ambient space, we observe the widest variety of cases: if c_H does not vanish identically there can either be one or no surface of constant Gauß curvature in the family. To a surface of constant Gauß curvature there exist either two parallel surfaces of constant mean curvature or two parallel surfaces with constant sum of their curvature radii²¹⁾. If no surface of constant Gauß curvature is present in the family there is either one surface of constant mean curvature, one surface with constant sum of the curvature radii or one surface of either type.

Viewing the family of parallel linear Weingarten surfaces as a Guichard net corresponds to the “correct” choice of the family parameter — t becomes an elliptic function (26) of a new parameter r . We already learned that the branch points of t have a geometric meaning (if they are real) for the family of Weingarten surfaces: they describe the infinity sphere and surfaces of constant

²¹⁾ It seems remarkable that in case of elliptic ambient space the *extrinsic* Gauß curvature of a surface has to be a positive constant in order to have parallel constant mean curvature surfaces while, in hyperbolic ambient space, this condition meets the *intrinsic* Gauß curvature.

mean curvature. The cross ratio of the branch points of t turns out to be real or to lie on the unit circle depending on whether $\varepsilon\sqrt{k}$ is real or imaginary; consequently, the underlying torus — on which the elliptic function is defined when viewed as a function of a complex variable — is a rectangular or a rhombic torus. Thus, if the type ε of the Weingarten surfaces a Guichard net is built of is known the type of the underlying torus corresponds to the type of the ambient space — the ambiguous case of cross ratio 1 (where the torus degenerates to a cylinder) corresponding to Euclidean space. Moreover, we can establish various relations between the cross ratio of the branch points of the elliptic function t and geometric quantities arising from the family of parallel Weingarten surfaces: for example, in case of cross ratio -1 (square torus) the zeros of c_K and c coincide, i.e. all surfaces of constant mean curvature present in the family are minimal, and the (extrinsic) Gauß curvature of any surface of constant Gauß curvature equals the ambient space's curvature.

We leave the complete analysis of the situation to the interested reader.

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